Proof of the necessity and sufficiency between the condition of Eq. (6) and that of Eqs. (7) and (8) is provided in Appendix I of Ref. 8. Notice that the condition of Eqs. (7) and (8) differs from the implicit assumption invoked in Ref. 4 (Note 11.2) (that W is nonsingular) in three respects. First, P_3^{-1} is not assumed to exist; use of the pseudoinverse $P_3^\dagger$ will properly handle any situation regarding P_3, whether or not zero eigenvalues are present. Second, the condition of Eq. (7) is such that the submatrices are established to be positive semidefinite rather than just a condition on the associated determinants, as in Eq. 2. Third, there is an additional criterion present on the nesting of null spaces[9] of P_3 and P_2; the condition of Eq. (8) must also be satisfied before the conclusion can be made that the P of Eq. (5) is positive semidefinite.

The condition of Eq. (8) fails to be satisfied for the numerical example of Eq. (4) since it reduces to

$$R \not\subset \mathcal{N}[P_2] \equiv \phi \tag{9}$$

thus enabling the correct conclusion to be drawn that the matrix of Eq. (3) is *not* positive semidefinite. When the condition of Eq. (8) is applied to the second example of Ref. 2 (Eq. 2) [identical to Eq. (3) here except that the parameter value a appears in the place of the zero in the third row and column], then the resulting

$$\phi \subset \mathcal{N}[P_2] \tag{10}$$

satisfies the required *nested subspace property* (for all nonzero values of a) and enables the correct conclusions to be drawn that the corresponding full matrix P is indeed positive semidefinite for $a \geq 1$, while failing to be so if $1 > a > 0$ [where the second condition of Eq. (7) correctly comes into play to reveal this lack of positive semidefiniteness in the latter case of $1 > a$].

III. Conclusion

Some prevalent misconceptions on how to test matrices for positive semidefiniteness (both theoretically and computationally) were reviewed. A simple counterexample revealed that a recently offered partitioned test for demonstrating the positive semidefiniteness of a matrix (with the potential of being applied stagewise to the higher dimensional matrices encountered in industrial applications) is flawed. A proper version of such a test was discovered, as historically developed by others in preparing to perform matrix spectral factorization (which involves matrices whose entries are polynomials or rational functions of a complex variable), but which is also valid in the simpler case here where the matrices of interest have constant numerical entries. The key difference between the incorrect and correct version of the partitioned test is that a condition involving the nesting of associated null spaces corresponding to two of the critical partitions must also be satisfied in order to properly conclude that the matrix is positive semidefinite.

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Estimating Projections of the Controllable Set

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Introduction

THE theory used to find controllable or reachable sets provides a useful tool in engineering design and analysis. For example, consider the following illustrative problem: A simple spring mass (k-m) system subject to a control force f is described by

$$m\ddot{y} + ky = f \tag{1}$$

where y is the displacement of the mass. Suppose that $k/m = 6/s^2$ and that a proportional, integral, derivative (PID) control loop of the form

$$f = m[r - (5y + 6 \int y \, dt + 6\dot{y})] \tag{2}$$

is put about the system so that the controlled dynamics are given by

$$\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = r \tag{3}$$

where r is a command input. The problem is to determine the maximum possible energy in the spring-mass system if it is initially at equilibrium, but for time $t > 0$, it is subject to an unknown but bounded command input. This problem may be solved by first finding those points in the three-dimensional state space (position-velocity-acceleration) that are reachable from the equilibrium point under the bounded command input (the reachable set) and then finding where on the reachable set the energy ($\frac{1}{2}m\dot{y}^2 + \frac{1}{2}ky^2$) is maximized. Note, however, that since only the energy is of interest, it would be sufficient to have knowledge of only the projection of the reachable set onto the position-velocity space. In this case, as indeed with many other similar problems, it would be useful to have a theory that could directly provide the projected controllable or reachable set.

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Any theory developed for controllable sets may, with proper modification, also be used to find reachable sets. Thus, to simplify matters in what follows, we will refer to controllable sets only. Furthermore, we will restrict the analysis to a dynamical control system of the form

$$\dot{x} = f(x,u) \tag{4}$$

where $x = [x_1,\ldots,x_n]^T$ is an n-dimensional state vector, u is a scalar control, and $f = [f_1,\ldots,f_n]^T$ is a c^1 function of the state x and control u. The dot denotes differentiation with respect to time. The control can be any piecewise continuous function of time that satisfies the constraint condition $u \in U$ with U defined by

$$U = \{u \in R \mid u_{\min} \leq u \leq u_{\max}\} \tag{5}$$

Associated with the system (1) is a target set defined by

$$\theta(x) = 0 \tag{6}$$

A fundamental problem in control theory is to find those points in state space where one can guarantee driving the system to the target. The set of all such points is called the controllable set.[1] The boundary of the controllable set may be found for problems of low dimension by directly applying the controllability maximum principle.[2,3] This principle has been used to find the boundaries of controllable sets for a number of problems of one,[4] two,[1,4,5] and three dimensions.[6–8]

However, use of the controllability maximum principle to define the controllable set is, for all practical purposes, limited to problems of three dimensions or less. Since we may be interested in only projection information, as noted in the foregoing example, it is of interest to consider using the controllability maximum principle to determine projections of higher-dimension controllable sets onto lower dimensions.

Projected Controllable Sets

Since the ordering of the components of the state vector is arbitrary, assume that the first few components define the space on to which the controllable set is to be projected. In particular, let these components be designated by the vector

$$p = [p_1,\ldots,p_q]^T \equiv [x_1,\ldots,x_q]^T \tag{7}$$

where $1 \leq q \leq n-1$. Let

$$z = [z_1,\ldots,z_{n-q}]^T \tag{8}$$

be a vector related to x through a nonsingular linear transformation matrix Q of the form

$$\begin{bmatrix} p \\ z \end{bmatrix} = Qx = \begin{bmatrix} I & 0 \\ Q_{21} & Q_{22} \end{bmatrix} [x] \tag{9}$$

where I is a $q \times q$ identity matrix and Q_{21} and Q_{22} are $(n-q) \times q$ and $(n-q) \times (n-q)$ matrices, respectively. If Q can be chosen such that the transformed state variables satisfy a system of equations of the form

$$\dot{z} = f_A(z,u) \tag{10}$$

$$w = g(z,u) \tag{11}$$

$$\dot{p} = f_C(p,w) \tag{12}$$

where w is a scalar "output" from Eq. (10) and f_A, g, and f_C are all c^1 functions of their arguments, then these equations may be used in various combinations to obtain both an underestimate and overestimate of the projected controllable set. Note that Eqs. (10) and (11) are decoupled from Eq. (12), and that Eq. (12) expresses the dynamics of the projected system (e.g., $p = [x_1, x_2]^T$) in terms of the projected state variables p and an output w as obtained from Eq. (11). Equation (10) expresses the remaining dynamics as a function of the remaining transformed variables and original input u.

The basic idea, supported by theorems to follow, is contained in Fig. 1. If the original system has an equivalent representation given by Eqs. (10–12), we may think of the input to system A as producing an output that, in turn, drives system C. We first find the domain of the output w by the methods explained next. If we then find the controllable set to the projection of the target in p space for system C with w as the input subject only to the domain just obtained, the resulting controllable set must be an overestimate of the projection of the actual controllable set (see theorem 1). We then find a closed-loop control law for u that will drive system A from any point in its controllable set to the projection of the target in z space. If this control law is then used to drive the input to system A whose output, in turn, drives system C, then the domain of attraction defined by the output of system C must be an underestimate of the projection of the actual controllable set (see Theorem 2).

Basic Theory

In order to prove the theorems, we will need three additional systems in addition to those depicted in Fig. 1. All five systems are summarized in Table 1.

The original system subjected to the control constraint [Eq. (5)] is assumed to have a nonempty controllable set, designated by C_x. Since one can drive the system from all points in C_x to the target, there must exist a feedback control law $\bar{u}(x)$ that will do the job.

System A-B is equivalent to the original system, but with a different state-variable representation. It is, however, subject to the same control constraint (5) that will produce a controllable set $\tilde{C}$. Again, there must exist a feedback control law $\bar{u}(z,p)$ that will drive this system from any point in $\tilde{C}$ to the target.

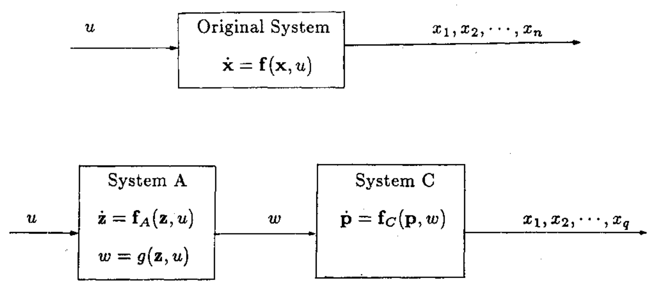

Fig. 1 Equivalent representation for a single-input control system.

Table 1 Systems used in proof of the theorems

System	Representation	State variables	Control variable	Set of interest	Control law
Original	$\dot{x} = f(x,u)$	x	$u \in U$	Controllable set C_x	$\bar{u}(x)$
A-B	$\dot{z} = f_A(z,u)$ $\dot{p} = f_B(p,z,u)$	z,p	$u \in U$	Controllable set $\tilde{C}$	$\bar{u}(z,p)$
C	$\dot{p} = f_C(p,w)$	p	$w \in W$	Controllable set C_p	$\bar{w}(p)$
A	$\dot{z} = f_A(z,w)$	z	$u \in U$	Controllable set C_z	$\bar{u}(z)$
A-C	$\dot{z} = f_A(z,u)$ $w = g(z,u)$ $\dot{p} = f_C(p,w)$	z,p	$u \in U$	Domain of attraction Obtained from $\bar{u}(z)$ $\tilde{D}$	$\bar{u}(z)$

The state variables of system C are just the first few state variables (projected variables) of the original system. Its control variable is w, and as a standalone system, its control is assumed to be subject to a constraint set defined by

$$W = \{ w \in R \mid w = g(z,u) \ \forall z \in C_z \quad \text{and} \quad u \in U \} \tag{13}$$

where C_z is the controllable set for system A to follow. Note that W is the domain of g over the range $C_z \times U$. The controllable set for this system is designated by C_p and let $\bar{w}(p)$ be a feedback control law that will drive system C from any point in C_p to the target.

The state variables of system A are the remaining transformed variables z. It is subject to the same control constraint set as the original system, resulting in a controllable set C_z. Let $\bar{u}(z)$ be a feedback control law that will drive system A from any point in C_z to the target.

The final system A-C is the equivalent system of Fig. 1. Its state variables are z and p with the control u subject to Eq. (5). The underestimate for the projection of C_x onto the p space will be obtained from the domain of attraction to the target set under the control law $\bar{u}(z)$.

To prove the first theorem, we first note that if $\text{proj}_z\tilde{C}$ is the projection of the controllable set for system A-B onto z space and C_z is the controllable set for system A, then

$$\text{proj}_z\tilde{C} \subseteq C_z \tag{14}$$

This follows since for any point $(p,z) \in \tilde{C}$, there exists a control $u(t)$ that will drive system A-B from (p,z) to the target, and if this same control is applied to system A from the point z, system A will also be driven to the target in z space since Eq. (10) is uncoupled from Eqs. (11) and (12).

If we now calculate a new constraint set W' defined by

$$W' = \{ w \in R \mid w = g(z,u) \ \forall z \in \text{proj}_z\tilde{C} \quad \text{and} \quad u \in U \} \tag{15}$$

it follows from Eqs. (13) and (14) that $W' \subseteq W$. If we now find the controllable set to system C with w constrained by Eq. (15), we obtain a set C_p' that, since W' is more restrictive than W, is related to C_p by

$$C_p' \subseteq C_p \tag{16}$$

Theorem 1: If proj_pC_x is the projection of the controllable set for the original system onto p space and C_p is the controllable set for system C subject to the control constraint $w \in W$, then

$$\text{proj}_pC_x \subseteq C_p \tag{17}$$

Proof: Let Ω' be the set of all continuous functions $w(\cdot)$ satisfying $w(t) \in W'$ for all $t > 0$ and C_p' be the controllable set for system C corresponding to $w(\cdot) \in \Omega'$. Let Ω'' be the set of all functions $w(t)$ obtained from Eqs. (10) and (11) for all piecewise continuous functions $u(\cdot)$ satisfying $u(t) \in U$ for $t > 0$ and which produce trajectories that satisfy $x(t) \in C_x$ for $t \geq 0$. Note that all functions Ω'' satisfy $w(t) \in W'$ for $t > 0$ because of definition (15). Let C_p'' be the controllable set for system C subject to $w(\cdot) \in \Omega''$. Since Ω'' is more restrictive than Ω' ($\Omega'' \subset \Omega'$), it follows that

$$C_p'' \subseteq C_p' \tag{18}$$

To complete the proof, we need to show that

$$\text{proj}_pC_x \subseteq C_p'' \tag{19}$$

Assume $\text{proj}_pC_x \not\subseteq C_p''$ and let p be a point in C_p'', but not in proj_pC_x. This point is controllable to the target by means of some $w(\cdot) \in \Omega''$. Let $u(t)$ be the control that generated $w(t)$. By means of this $u(t)$, we can drive p to the target while remaining in C_x, which is a contradiction to the assumption. It then follows from Eqs. (18) and (19) that

$$\text{proj}_pC_x \subseteq C_p' \tag{20}$$

Condition (17) then follows from Eqs. (16) and (20).

Theorem 1 provides one way of determining an overestimate for proj_pC_x. Alternate methods exist for obtaining overestimates for C_x (from which an overestimate of proj_pC_x could be obtained) based on Lyapunov methods[9,10] and decoupling techniques for linear systems.[11] However, none of these methods provide underestimate information, so it is not possible to judge the accuracy of the overestimate.

The following theorem provides a way for determining an underestimate for proj_pC_x.

Theorem 2: If proj_pC_x is the projection of the controllable set for the original system onto p space and $\text{proj}_p\tilde{D}$ is the projection of the domain of attraction onto p space for system A-C under the closed-loop control $\bar{u}(z)$, then

$$\text{proj}_p\tilde{D} \subseteq \text{proj}_pC_x \tag{21}$$

Proof: If $\text{proj}_p\tilde{D} \not\subseteq \text{proj}_pC_x$, then there must exist a point $(p,z) \in \tilde{D}$ with $p \notin C_x$ that can be driven to the target under the control law $\bar{u}(z)$. Since system A-C can be transformed to the state space of the original system by means of a nonsingular transformation [Eq. (9)], it follows that there must exist points not in C_x that can be driven to the target. This contradicts the assumption that C_x is the controllable set for the original system, and the theorem follows.

Application to Linear Systems

In order to use theorems 1 and 2 to estimate proj_pC_x, one must first be able to put the original system into the form of Eqs. (10–12). Wu[12] has shown that a transformation Q will always exist for linear input/output (I/O) systems of the form

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1\dot{y} + a_0y = bu \tag{22}$$

where $y^{(n)} = \mathrm{d}^ny/\mathrm{d}t^n$. By choosing $x_1 = y$, $x_2 = \dot{y}, \ldots, x_n = y^{(n-1)}$, this system is first put into a state-space representation of the form

$$\dot{x} = Ax + Bu \tag{23}$$

where A and B are in companion form. Under the Q transformation, Eq. (23) becomes

$$\begin{bmatrix} \dot{p} \\ \dot{z} \end{bmatrix} = QAQ^{-1}\begin{bmatrix} p \\ z \end{bmatrix} + QBu \tag{24}$$

and is of the form of Eqs. (10–12) if one chooses

$$w = z_1 \tag{25}$$

System C will have the first two eigenvalues and system A will have the rest.[12]

Example

Consider again the I/O system previously given. In order to find the reachable set for Eq. (3), we may instead find the controllable set for the retrosystem given by

$$\dddot{y} - 6\ddot{y} + 11\dot{y} - 6y = u \tag{26}$$

where $r = -u$, and the dot now denotes differentiation with respect to negative (retro) time. Assume that $|u| \leq 1$.

The following Q matrix

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 6 & -9 & 3 \end{bmatrix} \tag{27}$$

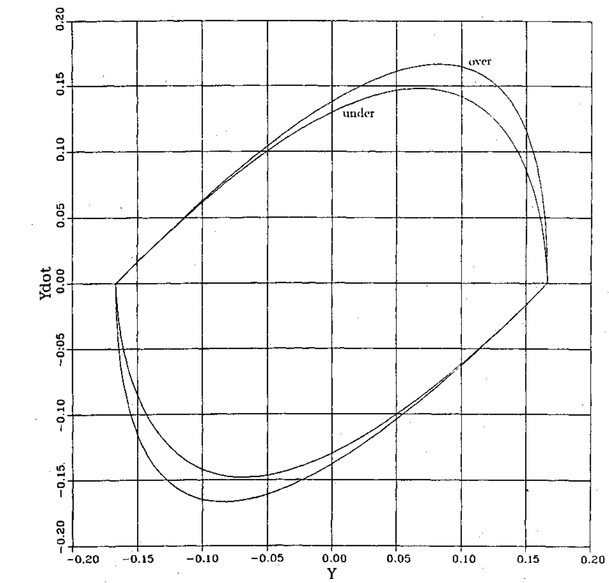

Fig. 2 Overestimate and underestimate for the projected controllable set to the origin for the three-dimension example.

transforms the equivalent state-space system (23) to

$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{z}_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 3 & 1/3 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ z_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} u \tag{28}$$

By setting $w = z_1$, we have a system in the form of Eqs. (10–12)

$$\left.\begin{aligned} \dot{z}_1 &= 3z_1 + 3u \\ w &= z_1 \end{aligned}\right\} \text{system } A \tag{29}$$

$$\left.\begin{aligned} \dot{p}_1 &= p_2 \\ \dot{p}_2 &= -2p_1 + 3p_2 + 1/3\ w \end{aligned}\right\} \text{system } C \tag{30}$$

We may now obtain both upper and lower estimates for the projected controllable set. The bounds W are obtained from the equilibrium solution to Eq. (29) under $u = +1$ and $u = -1$. We obtain

$$|w| \leq 1 \tag{31}$$

Now using w subject only to Eq. (31) as the input to system C, we must now find the control law for w that yields trajectories on the boundary of the controllable set for system C. A simple prescription for doing this is given by Gayek and Vincent.[13] In particular, with real eigenvalues for system C, we need only integrate backward from the equilibrium point corresponding to $w = -1$ using the control $w = +1$ and then integrate backward from the equilibrium point corresponding to $w = +1$ using $w = -1$. The two trajectories obtained will be the boundary of the controllable set for system C. The resulting controllable set labeled "over" in Fig. 2 is an overestimate for the actual projection of the true controllable set in accordance with theorem 1.

In order to obtain the underestimate, we must find a closed-loop control law $\bar{u}(z)$ as the input for u to drive both system A and system C together (i.e., the overall system). A closed-loop control law that will drive system A from any point in C_z to the origin is given by

$$\bar{u}(z) = \begin{cases} -kz_1 & \text{if } |kz_1| \leq 1 \\ -\operatorname{sgn}(kz_1) & \text{if } |kz_1| > 1 \end{cases} \tag{32}$$

for $k > 3$. The domain of attraction $\tilde{D}$ for the system A-C under this control law is obtained by integrating system A from any point in the neighborhood of the origin backward in time under $\bar{u}(z)$ until a boundary point of C_z is identified. At this point, the control is switched in sign and integration is continued until another boundary point is identified, etc. By plotting the resulting trajectories in p space, we are able to identify $\text{proj}_p\tilde{D}$, which in turn defines an underestimate of the true controllable set as illustrated in Fig. 2 with the curve labeled "under." In this case, the underestimate is identical to the actual projection.[12]

Summary

This Note extends the usefulness of the controllability maximum principle by demonstrating its ability to find estimates of projections of the controllable set onto a lower-dimension (i.e., two-dimensional) space. Three- or four-dimensional problems may now be "solved" with about the same difficulty as two-dimensional problems were previously solved. The potential exists for solving higher-dimension problems as well. The method is applicable to both linear and nonlinear systems, however, only for a class of linear systems has it been shown that a transformation matrix exists that allows for a direct application of the method.

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